Localization, completions and metabelian groups

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What do we want?

(1) Examples of maps between different f.g. residually nilpotent groups

$$G \to H$$

which induce isomorphisms $\widehat{G} \simeq \widehat{H}$.

(2) Given a group G, to describe (if possible) all para-G-groups.

(3) Determine the properties of groups which are common for all para-G-groups (homological, finite presentability, linearity etc).

A group G is metabelian if $[G, G]$ is abelian.

Suppose R is a commutative ring with unit, M is an R-module, and S a multiplicative set containing the unit, $1 \in R$. We denote the result of inverting elements of S by M_S. Specifically, consider the abelian group

$$M_S := MS^{-1} = (M \times S)/\sim$$

where

$$(x, s_1) \sim (y, s_2)$$

if there is an element $s \in S$ such that

$$(xs_2 - ys_1)s = 0.$$

We denote the element (x, s) by the notation $\frac{x}{s}$, and the group law for M_S is given by

$$\frac{x}{s_1} + \frac{y}{s_2} = \frac{xs_2 + ys_1}{s_1 s_2}.$$

M_S is an R-module via the scalar action

$$\frac{x}{s} r = \frac{xr}{s}.$$

Primary Invariants.

We assume that groups we consider are finitely generated.

Theorem. Suppose H is para-G.

- Let $S = 1 + \mathrm{ker}\{\mathbb{Z}[G_{ab}] \to \mathbb{Z}\}$. Then
$$S^{-1}[G,G] \cong S^{-1}[H,H].$$

- Let $R = \mathbb{Z}[G_{ab}]/Ann([G,G])$, where
$$Ann([G,G]) = \{r \in \mathbb{Z}[G_{ab}] \mid r \cdot m = 0 \text{ for all } m \in [G,G]\}.$$
Then the localized $\mathbb{Z}[G_{ab}] = \mathbb{Z}[H_{ab}]$-module and associated rings for G and H are isomorphic.

In analogy with Algebraic Geometry, we call the ring $\mathbb{Z}[G_{ab}]/Ann([G,G])$ the *coordinate ring of* G.

Examples.

Theorem. If G is a finitely presented, residually nilpotent, metabelian group, and the coordinate ring of G is a principal ideal domain, then any para-G group is isomorphic to G.

- The Lamplighter group

$$\mathbb{Z}/2 \wr \mathbb{Z} = \mathbb{Z}/2[t, t^{-1}] \rtimes \mathbb{Z}.$$

$$\langle a, t \mid a^2 = 1,\ [a, a^{t^i}] = 1,\ i \in \mathbb{Z} \rangle$$

- For $n \neq 2$, the group

$$\langle a, b \mid aba^{-1} = b^n \rangle$$

We have a group we call G_S which is defined by the following diagram (it is a "push-out of extensions"induced by localization $[G,G] \to [G,G]_S$):

$$\begin{tikzcd}
1 \arrow[r] & {[G,G]} \arrow[r] \arrow[d, tail] & G \arrow[r] \arrow[d, tail] & G_{ab} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & {[G,G]_S} \arrow[r] & G_S \arrow[r] & G_{ab} \arrow[r] & 1
\end{tikzcd}$$

Properties:
(1) G_S is Levine's localization of G;
(2) if H is para-G, then $H_S \simeq G_S$.

Telescope Theorem Given a residually nilpotent, metabelian group G, there is a sequence of groups

$$G^0 \subset G^1 \subset G^2 \subset \cdots \subset \cup G^k = G_S$$

and $G^k \cong G$ for all k.

Corollaries of Telescope Theorem:

(1) Let G and H be f.g. metabelian residually nilpotent. If H is para-G, then G is para H. That is we have an equivalence relation;

(2) if G and H are para-equivalent, then G is finitely presented iff H is;

(3) if G and H are para-equivalent, then G is polycyclic iff H is.

Some number theory!

Consider the ring of cyclotomic integers,

$$\mathbb{Z}[\zeta_n] \cong \mathbb{Z}[t, t^{-1}]/(\phi_n(t)),$$

where $\phi_n(t)$ is the n-th cyclotomic polynomial.

Let $G = \mathbb{Z} \ltimes \mathbb{Z}[\zeta_n]$, where the action of a generator t of $\mathbb{Z}$ on $\mathbb{Z}[\zeta_n]$ is multiplication by ζ_n. G is residually nilpotent if and only if $n = p^k$ for some prime p and positive integer k.

$D = \mathbb{Z}[\zeta_n]$ is a principal ideal domain for $n < 23$, and any group para-equivalent to $G = T \ltimes \mathbb{Z}[\zeta_{p^k}]$ is isomorphic to G for prime powers $p^k < 23$.

The first interesting case occurs for $n = 23$. In this case the following is a para-equivalence of non-isomorphic groups.

$$\mathbb{Z} \ltimes \left(2, \frac{1 + \sqrt{-23}}{2}\right) \subset \mathbb{Z} \ltimes \mathbb{Z}[\zeta_{23}].$$

Consider the number field, $\mathbf{Q}(\sqrt{d})$. The *ring of algebraic integers* in this number field, D, is the subring of all solutions to monic polynomials over the integers. This is:

$$D = \mathbb{Z}[\sqrt{d}] \text{ for } d \equiv 2, 3 \mod 4$$

$$D = \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] \text{ for } d \equiv 1 \mod 4.$$

There is an onto ring homomorphism $\mathbb{Z}[t, t^{-1}] \to D$, for $d < 100$ and $G = D \rtimes \mathbb{Z}$ is residually nilpotent when

$$d = 2, 3, 10, 13, 15, 23, 26, 29, 35, 53, 77, 82, 85$$

For $d = 2, 3, 13, 23, 29, 53,$ and 77, any group para-G is isomorphic to G.

For $d = 10, 15, 26, 35, 85$ there are two groups in each para-equivalence class.

For $G = \mathbb{Z}[\sqrt{82}] \rtimes \mathbb{Z}$, there are 4 groups in the para-equivalence class of G.

Classification Theorem

We call a submodule of $A \subset [G,G]_S$ an $\{S$-fractional submodule$\}$ if the inclusion induces $A_S \cong [G,G]_S$. We denote the set of S-fractional submodules of $[G,G]_S$ by

$$\mathcal{F}([G,G]_S).$$

An automorphism of G_S determines an automorphism of $[G,G]_S$, and therefore an action of $Aut(G_S)$ on $\mathcal{F}([G,G]_S)$. We term two fractional S-modules equivalent if such an induced automorphism of $[G,G]_S$ maps one onto the other.

Let $\mathcal{C}\ell(G) = \frac{\mathcal{F}([G,G]_S)}{Aut(G_S)}$.

$$\{Isomorphism\ classes\ of\ groups$$
$$para-equivalent\ to\ G\} \overset{1-1}{\leftrightsquigarrow} \mathcal{C}\ell(G)$$