COCYCLES, GALOIS THEORY AND AUTOMORPHIC FORMS

Hans Opolka

Abstract: This is a report about connections between central simple algebras, Galois representations and automorphic forms, with some emphasis on finite Heisenberg groups with Galois action.

§1. Regular crossed products and Galois representations

At first we describe a simple construction which leads to a connection between central simple algebras over a field which satisfies a certain cohomological condition and representations of the absolute Galois group of that field; see [$O3$] for a preliminary report about this connection..

Let k be a field with a separable algebraic closure $\overline{k}$. For every subextension K/k of $\overline{k}/k$ let $G_K = G(\overline{k}/K)$ denote the profinite Galois group of the extension $\overline{k}/K$ and denote by μ_K the group of roots of unity in K. For every positive integer m let μ_m denote the group of m-th roots of unity in $\overline{k}$. As was shown in [B], §6, Satz 10, every associative finite dimensional central simple k-algebra $\mathcal{A}$ such that the characteristic of k does not divide the exponent of $\mathcal{A}$ is similar to a crossed product $(K/k, c)$ where K/k is a finite Galois subextension of $\overline{k}/k$ - with Galois group $G = G(K/k)$ say - and where $c : G \times G \to \mu_m \hookrightarrow \mu_K$ is a Galois 2-cocycle for some positive integer m. Adopting a terminology from [B], §2, such a crossed product is called *regular*. Note that according to [B], §4, Satz 7, the embedding problem for G_k which arises from the group extension which is defined by the cocycle $c : G \times G \to \mu_m$ is solvable if and only if the similarity class of $(K/k, c)$ is trivial in the Brauer group $Br(k)$ of k. Many of the considerations in [B] which are formulated in terms of algebras and group extensions, especially those which are related to embedding problems, can be translated into simple cohomological terms; comp. e.g. [HO], especially section 1.

Assume now that the characteristic of k is 0 and that $(K/k, c)$ is a regular crossed product. Denote by $m = m(c)$ the order of c, i.e. the smallest positive integer j such that $c(s,t)^j = 1$ for all $s, t \in G = G(K/k)$. We will construct a set $\mathcal{R}(K/k, c)$ of isomorphism classes of continuous irreducible representations $R : G_k \to GL(n, \mathbb{C})$ - where G_k is regarded as a topological group with respect to the profinite topology and where $GL(n, \mathbb{C})$ is regarded as a topological group with respect to the discrete topology - under the assumption that the cohomology group $H^2(G_M, \mathbb{Q}/\mathbb{Z})$ is trivial for every finite abelian subextension M/k of $\overline{k}/k$. It is well known that this assumption holds if k is a local or global number field, see [T]; [$SE1$], §6; and - of course - it holds if k is a field of cohomological dimension ≤ 1; see [$SE4$], II, §3, especially 3.3 for examples. In order to construct $\mathcal{R}(K/k, c)$ put $G_m := G(K/k(\mu_m))$, $c_m :=$ restriction of c to $G_m \times G_m$; so $c_m : G_m \times G_m \to \mu_m$ is a *central* 2-cocycle with trivial action of G_m. Identify μ_m with the group of roots of unity of order m in $\mathbb{C}$. Let $T : G_m \to GL(n, \mathbb{C})$ be any continuous irreducible c_m-representation of G_m, comp.[M] for the terminology; so we have $T(s)T(t) = c_m(s,t)T(st)$ for all $s, t \in G_m$. It follows from the assumed triviality of $H^2(G_{k(\mu_m)}, \mathbb{Q}/\mathbb{Z})$ that T has a lifting, i.e. that there is a continuous irreducible linear representation $D : G_{k(\mu_m)} \to GL(n, \mathbb{C})$ such that the corresponding projective representations $\overline{D}$ and $\overline{T}$ coincide, comp. e.g. $[SE1]$, §6. Denote by $\mathcal{R}_D$ the set of all isomorphism classes of continuous irreducible linear representations R of G_k of finite degree such that the restriction of R to $G_{k(\mu_m)}$ contains D as an irreducible constitutent, and let $\mathcal{R}(K/k, c)$ denote the union of all the sets $\mathcal{R}_D$ where D is any continuous linear representation of $G_{k(\mu_m)}$ of finite degree which lifts an irreducible c_m-representation of G_m. Using the Clifford-Mackey theory, comp. $[CL]$ and $[M]$, one proves that the degree of every $(R) \in \mathcal{R}(K/k, c)$ divides the degree $(K : k)$. Finally, for every similarity class a of central simple k-algebras denote by $\mathcal{R}(k, a)$ the union of all the sets $\mathcal{R}(K/k, c)$ where $(K/k, c)$ is any regular crossed product representing a.

§2. Regular crossed products and Galois representations in the case of number fields

Let k be a number field. A continuous linear or projective representation D of G_k over $\overline{k}$ of finite degree is said to be unramified outside a given finite set of places S of k, if for all places of k which do not belong to S the corresponding inertia subgroups are contained in the kernel of D. As is well known, the following result can be deduced - by applying the nonabelian version of the "Führerdiskriminantenproduktformel" $[S2]$, VI, §3, Cor.2, p. 104 -

from a result of I. Schur $[S]$ which shows that the order of a finite subgroup G of $GL(n, \mathbb{C})$ such that the traces of all elements of G belong to a fixed subfield $E \subset \mathbb{C}$ of finite degree $d = (E : \mathbb{Q})$ divides a certain number which depends only on n and d

and from a variant of the well known result of Hermite and Minkowski, see e.g. $[K]$, Satz 2.13.6, S. 57, according to which there are only finitely many number fields with a given discriminant;

comp. e.g. $[J]$, Satz (1.5), p. 10.

(2.1) **Proposition** *Let k be a number field, let S be a finite set of places of k, let n be a positive integer and let $E \subset \mathbb{C}$ be a subfield of finite degree $(E : \mathbb{Q})$. Then there are only finitely many isomorphism classes of continuous linear representations $R : G_k \to GL(n, \mathbb{C})$ such that R is unramified outside S and such that all values of the character of R belong to E.*

In view of this result it seems worthwhile to investigate for a number field k rationality and ramification properties of representations of G_k which are constructed from regular crossed products as above. Similar investigations for related classes of representations are contained in $[J]$; see §4 below for more details.

Assume that k is a number field and that $\mathcal{A}$ is a central simple algebra over k. Denote by $S_{\mathcal{A}}$ the finite set of places of k at which $\mathcal{A}$ does not split. Let $(K/k, c)$ be a regular crossed product which is similar to $\mathcal{A}$. Denote by $S_{K/k}$ the finite set of all places which are ramified in K/k. Since all values of c are roots of unity it follows from the local theory of central simple algebras, see e.g. $[D]$, VII, §2, especially Satz 3, p.112, that $\mathcal{A}$ splits at all places which are unramified in K/k. So we have $S_{\mathcal{A}} \subset S_{K/k}$. It was observed by Hasse $[H]$, see also $[D]$, VII, Satz 4, S. 118, that there is a smallest multiple $g = g(\mathcal{A})$ of the exponent $\exp(\mathcal{A})$ of $\mathcal{A}$ such that $k(\mu_g)$ is a splitting field of $\mathcal{A}$; namely, by the local theory of central simple algebras and by the local-global principle for central simple algebras, see $[D]$, VII, §5, Satz 1, p. 117, g is the smallest positive multiple of the exponent of $\mathcal{A}$ such that the local degrees $(k_v(\mu_g) : k_v)$ are divisible by $\exp(\mathcal{A})$ for all $v \in S_{\mathcal{A}}$. Let $m = m(c)$ denote the order of c. Define the *cyclotomic index* $\widetilde{g} := \widetilde{g}(K/k, c)$ of $(K/k, c)$ by $l.c.m.(m(c), g(\mathcal{A}))$ if m is odd and by $l.c.m.(4.m(c), g(\mathcal{A}))$ if m is even. Using the profinite version of the exact Hochschild-Serre sequence, see e.g. $[SH]$,chapter II, §4, and results in $[P]$, section 2, one can prove

(2.2) **Proposition** *There is* $(R) \in \mathcal{R}(K/k, c)$ *such that all values of the character of* R *belong to the field* $\mathbb{Q}(\mu_{\exp(G(K/k)).\widetilde{g}(K/k,c)})$.

In order to obtain representations in $\mathcal{R}(K/k, c)$ with restricted ramification we introduce certain statements which are known to be true in special cases. For every positive integer e denote by S_e the set consisting of all infinite places of k and of all places of k which divide e, and for every finite set of places S of k let k_S/k denote the maximal Galois subextension of $\overline{k}/k$ which is unramified outside S.

(2.3) *Let* q *be a prime number, let* S *denote a finite set of places of* k *which contains* S_q *and let* $L(S, q)$ *denote the statement that the cohomology group* $H^2(G(k_S/k), \mathbb{Q}_q/\mathbb{Z}_q)$ *vanishes for every finite abelian subextension* M/k *of* k_S/k. *Furthermore, denote by* $L(k)$ *the statement that* $L(S, q)$ *holds for every prime number* q *and every finite set of places* $S \supset S_q$.

It is known that statement $L(k)$, which is related to the Leopoldt-conjecture, is true for $k = \mathbb{Q}$; see $[BR]$ in connection with $[MK]$.

Assume that S is a finite set of places of k containing $S_m \cup S_{K/k}$. Then if statement $L(k)$ is true there is a smallest multiple $l = l(K/k, c)$ of m with the same prime divisors as m such that the central embedding problem for $G(k_S/k(\mu_m))$ which is defined by the cocycle class $(c_m) \in H^2(G_m, \mu_m)$ is weakly solvable with respect to l, i.e. the central embedding problem for $G(k_S/k(\mu_m))$ corresponding to the image of (c_m) under the homomorphism (of cohomology groups with respect to the trivial group action)

$$H^2(G_m, \mu_m) \to H^2(G_m, \mu_l) \text{ induced by } \mu_m \hookrightarrow \mu_l$$

is solvable, comp. $[NO]$.

For any continuous representation R of G_k denote by S_R the set of all places of k which are ramified in the fixed field of the kernel of R.

(2.4) **Proposition** *If the statement $L(k)$ is true then there is $(R) \in \mathcal{R}(K/k,c)$ such that $S_R \subset S_m \cup S_{K/k}$ and such that all values of the character of R belong to $\mathbb{Q}(\mu_{\exp(G).l(K/k,c)})$.*

§3. Finite Galois modules, Heisenberg groups and regular crossed products

If $f : A \times A \to C$ is a central 2-cocycle on an abelian group A with values in an abelian group C then, as was observed in $[IM]$, §1, p. 132, the mapping $\omega_f : A \times A \to C$, $\omega_f(x,y) := f(x,y)/f(y,x)$ for all $x, y \in A$, is a bimultiplicative symplectic pairing; we call it *the symplectic pairing associated with* f.

Let k be a number field. Let A be a finite continuous G_k-module which is equipped with a G_k-equivariant central 2-cocycle $f : A \times A \to \mu_m$ such that the associated symplectic pairing ω_f is nondegenerate. Denote by $H(A,f)$ the central group extension defined by f, the so called *Heisenberg group corresponding to* $(A,\ f)$; $H(A,f)$ becomes a G_k-group by defining $s((\alpha,x)) := (s(\alpha), s(x))$ for all $\alpha \in \mu_m, x \in A$ and $s \in G_k$. The exact sequence of G_k-groups

$$1 \to \mu_m \to H(A,f) \to A \to 1$$

yields a coboundary map

$$H^1(G_k,A) \xrightarrow{\delta} H^2(G_k,\mu_m),$$

and therefore every cocycle class $(\alpha) \in H^1(G_k,A)$ defines a unique element in the Brauer group $Br(k)$ of k which can be represented by a regular crossed product $(K_{(\alpha)}/k,c)$, where $K_{(\alpha)}$ is a finite Galois splitting field of (α) and where $c = c_{(\alpha),f} : G_{(\alpha)} \times G_{(\alpha)} \to \mu_m \subset \mu_{K_{(\alpha)}}$ is a Galois 2-cocycle on $G_{(\alpha)} := G(K_{(\alpha)}/k)$ all of whose values belong to μ_m. Put $\mathcal{R}_{(\alpha),f} := \mathcal{R}(K_{(\alpha)}/k,c)$. Denote by K_A the fixed field of the kernel of the action of G_k on A. We assume

(3.1) $\quad \rho := res^{G_k}_{G_{K_A}}((\alpha)) \in Hom(G_{K_A},A)^{G(K_A/k)}$ *is surjective.*

It is easily seen that there is $(\alpha) \in H^1(G_k,A)$ satisfying (3.1) provided $H^2(G(K_A/k),A)$ vanishes. In fact, according to $[IK]$ there is a surjective solution ϕ of the embedding problem for G_k which is defined by the semidirect product of the $G(K_A/k)$-module A with $G(K_A/k)$. The restriction of ϕ to G_{K_A} yields a surjective $\rho \in Hom(G_{K_A},A)^{G(K_A/k)}$. Moreover, if $H^2(G(K_A/k),A)$ vanishes, the exact Hochschild-Serre sequence shows that there is $(\alpha) \in H^1(G_k,A)$ such that $\rho = res^{G_k}_{G_{K_A}}((\alpha))$. The construction of the set $\mathcal{R}_{(\alpha).f}$ shows that there is some $(R) \in \mathcal{R}_{(\alpha),f}$ from which the action of G_k on A can be reconstructed.

A variant of this construction is obtained by starting from a nondegenerate G_k-equivariant nondegenerate bimultiplicative pairing $f : A \times B \to \mu_m$ of finite continuous G_k-modules and by using the cup product

$$H^1(G_k,A) \times H^1(G_k.B) \to H^2(G_k,\mu_m) \hookrightarrow Br(k)$$

whose image consists of similarity classes of central simple k-algebras which can be represented by regular crossed products. This approach seems to be more adapted to the situation where f is the Poincaré duality pairing of finite level étale cohomology groups of a smooth projective variety defined over k.

Examples of G_k-modules A with trivial $H^2(G(K_A/k), A)$ arise naturally in the theory of elliptic curves; see $[SE3]$, especially examples 5.5.6 and 5.5.7, in connection with a result of Sah in group cohomology; comp. e.g. $[LG]$, chapter V. The case of elliptic curves is also discussed in more detail in §4 below, example 2.

Representations similar to those in $\mathcal{R}_{(\alpha),f}$ have been constructed in $[O1]$ by a different method and were investigated further in $[J]$. The algebraic framework is familiar from the construction of representations of Heisenberg groups and the associated metaplectic groups; comp. e.g. $[W]$ and also $[Z]$.

§4. Examples

Let k be a number field. In this section we describe various examples which illustrate the constructions above.

(1) Central pairs and Galois representations *;* comp. $[O2]$ for a related construction.

Let A be a finite abelian group and let $f : A \times A \to k^*$ be a central 2-cocycle. (A, f) is called a *central pair for* k.

As can be seen from E. Artin's construction of Clifford algebras in $[A]$,chapter V, section 4, p. 186ff, central pairs arise naturally in the theory of quadratic forms.

Another example of a central pair for the cyclotomic field $k = \mathbb{Q}(\mu_m)$ is obtained as follows. Assume that $t : A \times A \to \mu_m$ is a central 2-cocycle such that the associated symplectic pairing $\omega = \omega_t$ is nondegenerate. Let g denote the conductor of A, i.e. g is the smallest positive integer h such that there is an epimorphism $G(\mathbb{Q}(e^{2\pi i/h})/\mathbb{Q}) \to A$. For every character χ of A, viewed as a character of $G(\mathbb{Q}(e^{2\pi i/g})/\mathbb{Q})$, denote by $\tau(\chi)$ the corresponding Gaussian sum; for the terminology and elementary results on Gaussian sums comp. $[LE]$, §2. Furthermore, for every $x \in A$ let χ_x denote the character of A given by $\chi_x(y) := \omega(x, y)$, $y \in A$. Then (A, f), where

$$f : A \times A \to k^*, \;\; f(x, y) := t(x, y)\tau(\chi_x)\tau(\chi_y)\tau(\chi_x\chi_y) \text{ for all } x, y \in A,$$

is a central pair.

We assume now that k is a number field.

Let (A, f) be a central pair for k with the following properties:

(a) The symplectic pairing $\omega_f : A \times A \to k^*$ associated with f is nondegenerate; so especially all values of ω_f belong to μ_m where m is the exponent of A and $\mu_m \subset k^*$.

(b) The central pair (A, f) is *full*, which means that the following conditions (i) and (ii) hold:

(i) There is a map $\alpha_f : A \to \overline{k}^*$ such that $\alpha_f(x)^{ord(x)} = \prod_{j=1}^{ord(x)} f(x, x^j)$ for all $x \in A$

(ii) The degree of every $\alpha_f(x)$, $x \in A$, over k is the order $ord(x)$ of x, and the degree of the extension k_f/k which is generated over k by all $\alpha_f(x)$, $x \in A$, is the order of A.

Assume that (A, f) is a full central pair over the number field k. Then if we consider A as a trivial G_k-module, the composition of maps

$$\alpha : G_k \xrightarrow{\beta} \widehat{A} \xrightarrow{\gamma} A,$$

where $\beta(s)(x) := s(\alpha_f(x))/\alpha_f(x)$ for all $s \in G_k$, $x \in A$,and where for every $\lambda \in \widehat{A}$ the element $\gamma(\lambda) := x_\lambda \in A$ is such that $\lambda(y) = \omega_f(x_\lambda, y)$ for all $y \in A$, defines a surjective homomorphism

$$(\alpha) \in H^1(G_k, A) = Hom(G_k, A)$$

with the property $k_{(\alpha)} = k_f$. Let $f_0 : A \times A \to \mu_m$ denote a central 2-cocycle such that $\omega_f = \omega_{f_0}$. Then by the general construction in §3 the set $\mathcal{R}_{(\alpha),f_0}$ is defined. One can prove

(4.1) **Proposition** *The character group $\widehat{G}_k$ acts transitively on $\mathcal{R}_{(\alpha),f_0}$. Every $(R) \in \mathcal{R}_{(\alpha),f_0}$ has degree $|A|^{1/2}$. If statement $L(k)$ holds then there is $(R) \in \mathcal{R}_{(\alpha),f_0}$ such that $S_R \subset \{v : v$ divides m, v divides $a_f(x)$ for all $x \in A$, v divides $\infty\}$, and such that all values of the character of R belong to $\mathbb{Q}(\mu_l)$ where $l = l(K_{(\alpha)}/k, c_{(\alpha),f_0})$.*

Moreover, following $[O2]$, the results on automorphic induction in $[AC]$ imply the following proposition ($\mathbb{A}_K$ denotes the adele ring of a number field K).

.

(4.2) **Proposition** *Assume that the exponent of A is a prime number. Then every $(R) \in \mathcal{R}_{(\alpha),f_0}$ is cuspidal automorphic in the sense of $[L]$. More precisely, the cuspidal automorphic representation corresponding to $(R) \in \mathcal{R}_{(\alpha),f_0}$ is automorphically induced in the sense of $[AC]$ by a continuous character λ of $GL(1, \mathbb{A}_M)$ of finite order where $M \subset k_f$ is the fixed field corresponding to a maximal ω-isotropic subgroup of A under the isomorphism $G(k_f/k) \cong A$ which is induced by the epimorphism $\alpha : G_k \to A$ constructed above; and the character λ corresponds under the Artin map $GL(1, \mathbb{A}_M) \to G_M^{ab}$ (see $[AT]$) to a continuous character $\widetilde{\lambda}$ of G_M such that the restriction of $\widetilde{\lambda}$ to G_{k_f} is the central character γ_R of (R), i.e. γ_R is the unique irreducible constituent of the restriction of R to G_{k_f}.*

In the special case $A \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ the classical modular forms corresponding to the cuspidal automorphic representations include examples constructed by E. Hecke [HE] from indefinite binary quadratic forms; see [$O2$], section 4, and the literature mentioned there. Moreover for general A it can be shown that there is $(R) \in \mathcal{R}_{(\alpha),f_0}$ such that R and the corresponding cuspidal automorphic representation can be decomposed as an outer tensor product of so called hyperbolic representations where the underlying A´s are of type $\mathbb{Z}/m \times \mathbb{Z}/m$.

(2) Elliptic curves and Galois representations; comp. [$O1$] , [J].

.Let X be an elliptic curve defined over k. For any positive integer m denote by X_m the kernel of the multiplication by m homomorphism $X(\overline{k}) \overset{m}{\to} X(\overline{k})$. The Weil-pairing $\omega = \omega_m : X_m \times X_m \to \mu_m$ is a nondegenerate bilinear G_k-equivariant symplectic pairing, see e.g. [C] or [LG] . Let $f : A \times A \to \mu_m$ be a central 2-cocycle such that $f(x,y)/f(y,x) = \omega(x,y)$, $x, y \in A$. Assume that f is G_k-equivariant. Every k-rational point $P \in X(k) \setminus mX(k)$ defines an element $\Delta(P) \in H^1(G_k, X_m)$, $\Delta(P)(s) := s(Q) - Q$ for all $s \in G_k$, where $Q \in X(\overline{k})$ is such that $mQ = P$. If the restriction $\Delta(P)_{|G_{k(X_m)}} \in Hom(G_{k(X_m)}, X_m)^{G(k(X_m)/k)}$ is surjective, then, according to the above construction, the rational point P defines the set $\mathcal{R}_{P,f} := \mathcal{R}_{\delta(\Delta(P)),f}$ of isomorphism classes of continuous irreducible representations of G_k. Kummer theory for elliptic curves as developed in [BK] and [LG] , chapter V, yields examples with surjective $\Delta(P)_{|G_{k(X_m)}}$. We remark that if statement $L(k)$ holds then the set $\mathcal{R}_{P,f}$ contains the isomorphism class of a representation which is unramified outside the finite set of places which consists of all places above m and infinity and all places where the elliptic curve has bad reduction; comp. [J], sections 4.2-4.4.

The construction of an odd 2-dimensional Galois representation of octahedral type of Artin-conductor 59^2 in [HA] makes also use - at least implicitly - of an elliptic curve, namely $X : y^2 = x^3 + 2x - 1$. And the construction and thorough investigation of odd 2-dimensional Galois representations of $G_{\mathbb{Q}}$ of octaedral type in [BF] and [BU] is based on elliptic curves X over $\mathbb{Q}$ and nontrivial elements in $H^1(G_{\mathbb{Q}}, X_2)$ which are interpreted in terms of 2-coverings of X . For the theory of m-coverings of elliptic curves see [BS] and [C] . It offers a more geometric view of our construction.

References

[A] E. Artin: Geometric algebra, Interscience, New York, 1957

[AT] E. Artin, J. Tate: Class field theory, Benjamin, New York, 1967

[AC] J. Arthur, L. Clozel: Simple algebras, base change and the advanced theory of the trace formula, Annals of Mathematics Studies 120, Princeton University Press, New Jersey, 1989

[BK] M.I. Bashmakov: The cohomology of Abelian varieties over a number field, Russian Mathematical Surveys, 27, 1972, 2, 25-70

[B] R. Brauer: Über die Konstruktion der Schiefkörper, die von endlichem Rang in bezug auf ein gegebenes Zentrum sind, JRAM, 168, 1932, 44-64

[BR] A. Brumer: On the units of algebraic number fields, Mathematica, 14, 1967, 121-124

[BF] P. Bayer, G. Frey: Galois representations of octahedral type and 2-coverings of elliptic curves, Mathematische Zeitschrift, 207, 1991, 395-408

[BU] M. Bungert: Konstruktion von Modulformen niedrigen Gewichts, Preprint Serie, Institut für experimentelle Mathematik Essen, 12, 1990

[BS] B.J. Birch, H.P.F. Swinnerton-Dyer: Notes on elliptic curves I, JRAM, 212, 1963, 7-25

[C] J.W.S. Cassels: Diophantine equations with special reference to elliptic curves, J. London Math. Soc., 41, 1966, 193-291

[CL] A.H. Clifford: Representations induced in an invariant subgroup, Annals of Mathematics, 38, 1937, 533-550

[D] M. Deuring: Algebren, Springer Verlag, Berlin, 1935

[HA] K. Haberland: Perioden von Modulformen einer Variablen und Gruppenkohomologie III, Mathematische Nachrichten, 112, 1983, 297-315

[H] H. Hasse: Die Struktur der R. Brauerschen Algebrenklassengruppe über einem algebraischen Zahlkörper, Math. Ann., 107, 1933, 731-760

[HE] E. Hecke: Über einen Zusammenhang zwischen elliptischen Modulfunktionen und indefiniten quadratischen Formen, in: Mathematische Werke, Vandenhoeck und Ruprecht, Göttingen, 1959, No. 22, 1925, pp.418-427

[HO] K. Hoechsmann: Zum Einbettungsproblem, JRAM (Crelle Journal), 229, 1968, 81-106

[IK] M. Ikeda: Zur Existenz eigentlicher galoisscher Körper beim Einbettungsproblem, Abh. Math. Seminar der Univ. Hamburg, 24, 1960, 126-131

[IM] N. Iwahori, H. Matsumoto: Several remarks on projective representations of finite groups, Journal of the Faculty of Science of the University of Tokyo, Sect. I, 10, 1964, 129-146

[J] F. Jonas: Galoisdarstellungen und Einbettungsprobleme mit beschränkter Verzweigung, Dissertation, Universität Göttingen, 1991

[K] H. Koch: Zahlentheorie, Vieweg Verlag, Braunschweig/Wiesbaden, 1997

[LG] S. Lang: Elliptic curves diophantine analysis, Springer, Berlin, 1978

[L] R.P. Langlands: Problems in the theory of automorphic forms, in: Lectures in modern analysis and applications, LNM 170, Springer Verlag, New York, 1970

[LE] H.W. Leopoldt: Zur Geschlechtertheorie in abelschen Zahlkörpern, Math. Nachrichten, 9, 1953, 351-362

[M] G. W. Mackey: Unitary representations of group extensions I, Acta Mathematica, 99, 1958, 265-311

[MK] K. Miyake: Leopoldt kernels and central extensions of numbert fields, Nagoya Math. J., 120, 1990, 67-76

[NO] Q. D. Th. Nguyen, H. Opolka: Numerical invariants of central embedding problems, Journal of Number Theory, 52, 1995, 7-16

[$O1$] H. Opolka: Galoisdarstellungen und Galoiskohomologie von Zahlkörpern, Schriftenreihe des Mathematischen Instituts der Universität Münster, 2. Serie, Heft 27, Februar 1983

[$O2$] H. Opolka: Central pairs, Galois theory and automorphic forms, Algebras and Representation Theory, 6, 2003, 449-459

[$O3$] H. Opolka: Central simple algebras and Galois representations,
http://digibib.tu-bs.de/?docid=00028816, July 2009

[P] G. Poitou: Conditions globales pour des problèmes de plongement à noyau abélien, Annales de L'Institut Fourier, XXIX, 1979, 1-14

[S] I. Schur: Über eine Klasse von endlichen Gruppen linearer Substitutionen, Nr. 6 der Ges. Abh., Springer Verlag, Berlin, 1973

[$SE1$] J.P. Serre: Modular forms of weight one and Galois representations, in: A. Fröhlich (ed.): Algebraic number fields, Academic Press, New York, 1977; pp. 193-268

[$SE2$] J.P. Serre: Local fields, Springer Verlag, New York, 1979

[$SE3$] J.P. Serre: Propriete galoisiennes des points d'ordre fini des courbes elliptiques, Inventiones Mathematicae, 15, 1972, 259-331

[$SE4$] J. P. Serre: Galois cohomology, Springer Verlag, Berlin, 1997

[SH] S.S. Shatz: Profinite groups, arithmetic and geometry, Annals of Mathematics Studies, Princeton University Press, 1972

[T] J. Tate: Duality theorems in Galois cohomology over number fields, Proceedings of the International Congress of Mathematicians, Stockholm, 1962, 288-295

[W] A. Weil: Sur certains groupes d'operateurs unitaires, Acta Mathematica, 111, 1964, 143-211

[Z] E. W. Zink: Weil Darstellungen und lokale Galoistheorie, Mathematische Nachrichten, 92, 1979, 265-288