Hyperbolicity of Generic High-Degree Hypersurfaces

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- For the parallelism, the existence of nontrivial holomorphic map f from $\mathbb{C}$ to X is parallel to the finiteness of number of rational points in X.
- Maximum R (known as *Schottky radius*) for holomorphic map f_R from disk of radius R to X (after normalization is derivative of f_R at the center is parallel to the bound on the number of rational points in X.

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- Done by lifting holomorphic map to $\tilde{X}$ defined by ${z_{n+1}}^\delta - f(z_0, \cdots, z_n)$ in $\mathbb{P}_{n+1}$.

Version with Slowly Coefficients (Not Yet Proved in General Form)

- *Functional Relation with Slowly Varying Coefficients.* Let $a_{\nu_0 \cdots \nu_n}(\zeta)$ for $\nu_0 + \cdots + \nu_n = \delta$ be entire functions on $\mathbb{C}$ without common zeroes

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- Not yet proved in this form.

Version with Slowly Coefficients (Not Yet Proved in General Form)

- *Functional Relation with Slowly Varying Coefficients.* Let $a_{\nu_0\cdots\nu_n}(\zeta)$ for $\nu_0+\cdots+\nu_n=\delta$ be entire functions on $\mathbb{C}$ without common zeroes such that for some $\zeta\in\mathbb{C}$ the point $\left(a_{\nu_0\cdots\nu_n}(\zeta)\right)_{\nu_0+\cdots+\nu_n=\delta}$ of $\mathbb{P}_{\binom{n+\delta}{\delta}-1}$ does not belong to $Z_{\delta,n}$. Then there cannot exist entire functions $\varphi_0,\cdots,\varphi_n$ (not all ratios algebraic) on $\mathbb{C}$ without common zeroes such that

$$\sum_{\nu_0+\cdots+\nu_n=\delta} a_{\nu_0\cdots\nu_n}{\varphi_0}^{\nu_0}\cdots{\varphi_n}^{\nu_n}\equiv 0$$

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$$\lim_{r\to\infty} \frac{1}{T(r,\varphi)} \int_{\rho=0}^{r} \frac{d\rho}{\rho} \int_{\Delta_\rho} \varphi^* \eta.$$

- We now return to discussion of hyperbolicity of hypersurfaces.

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Rolf Nevanlinna, Zur Theorie der Meromorphen Funktionen. *Acta Math.* **46**, 1–99 (1925).

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- Logarithm derivative lemma applied to $d^\nu \log F_j \circ \varphi$ makes the average of $\log^+|\varphi^*\omega|$ on $|\zeta| = r$ dominated by positive constant times $\log T(\varphi, r)$, giving a contradiction.

Vanishing of Pullback of Jet Differential to Submanifold

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Forgetful Map Keeping Differentials and Forgetting Position

- The k-jet space $J_k(A)$ of A is a trivial bundle with fiber $\mathbb{C}^{N_k}$ generated by ${\partial_{z_1}}^{\nu_1} \cdots {\partial_{z_n}}^{\nu_n}$ with $1 \leq \nu_1 + \cdots + \nu \leq k$.

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- The k-jet space $J_k(A)$ of A is a trivial bundle with fiber $\mathbb{C}^{N_k}$ generated by ${\partial_{z_1}}^{\nu_1} \cdots {\partial_{z_n}}^{\nu_n}$ with $1 \leq \nu_1 + \cdots + \nu \leq k$.
- Forgetful map $\pi_k : J_k(A) = A \times \mathbb{C}^{N_k} \to \mathbb{C}^{N_k}$ is simply the natural projection onto the second factor.

$$\begin{array}{ccccc} Z_k = \overline{\mathrm{Im}\,(d^k\varphi)} & \hookrightarrow & J_k(A) = A \times \mathbb{C}^{N_k} & \stackrel{\pi_k}{\to} & \mathbb{C}^{N_k} \\ & & \sigma \downarrow & & \\ & & A & & \end{array}$$

- Good choice of P possible if and only if the generic fiber of the restriction of π_k to Z_k has finite fiber (for sufficiently large $k \geq k_n$).
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  the logarithmic derivative lemma corresponds to the following step.

- For $x_1, \cdots, x_m \in X$ rational, the quadratic property of the Neron-Tate height ($\|\cdot\|$ with inner product $\langle\cdot,\cdot\rangle$) is used to make the height of $x = (x_1, \cdots, x_m)$ small relative to the $\mathbb{Q}$-line bundle

$$L := -\varepsilon \sum_{i=1}^{m} s_i^2 \mathrm{pr}_i^*(L) + \sum_{i=1}^{m-1} (s_i x_i - s_{i+1} x_{i+1})^*(L)$$

  over A^m, ample over X^m

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- when x_i is chosen to make $\langle x_i, x_{i+1} \rangle \geq (1 - \frac{\varepsilon}{2}) \|x_i\| \|x_{i+1}\|$
- so that the Neron-Tate height of $s_i x_i - s_{i+1} x_{i+1}$ is $\left(s_i \sqrt{h_i} - s_{i+1} \sqrt{h_{i+1}}\right)^2 + \frac{\varepsilon}{2}$ plus s_i^2 times a bounded factor from the difference between the Neron-Tate height and the height with respect to L.

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- The quadratic property of the Neron-Tate height corresponds to the use of the zero curvature of the abelian variety A.
- What is unknown is how to use negative curvature directly in number theory without using embedding into an abelian variety of zero curvature

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- In Bloch's case a good choice of polynomial P of differentials $d^\nu w_j$, instead of dx, is used to get the required additional vanishing order.

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- We cannot use $0 = dR = R_x dx + R_y dx + R_a da$ to construct a holomorphic 1-form on $\mathcal{C}$,
- unlike the situation $0 = dR = R_x dx + R_y dx$ on C_a which enables us to divide by R_y to get ω_a.

$(n-1)$-Jet Differential on Hypersurface from Vanishing of Pullback of Low-Pole Order Jet Differential on $\mathbb{P}_n$

- Let X be a generic nonsingular hypersurface of degree δ in $\mathbb{P}_n$ defined by a polynomial $f(x_1, \cdots, x_n)$ of degree δ in the affine coordinates $x_1, \cdots, x_n$ of $\mathbb{P}_n$.

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- Then there exists an explicit positive number $A = A(n, \varepsilon, \varepsilon')$ depending only on n, ε, and ε' such that
- for $\delta \geq A$ and any nonsingular hypersurface X in $\mathbb{P}_n$ of degree δ there exists a non identically zero $\mathcal{O}_{\mathbb{P}_n}(-q)$-valued holomorphic $(n-1)$-jet differential ω on X represented by $\frac{Q}{f_{x_1} - 1}$,

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- If $m_0 + 2m < \delta$, then Q is not identically zero on the space of k-jets of X.

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- In our case we introduce a new technique of *slanted vector fields* to generate enough such jet differentials.

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- Fiber of projection $\mathbb{P}_n \times \mathbb{P}_{\binom{n+\delta}{\delta}-1} \to \mathbb{P}_{\binom{n+\delta}{\delta}-1}$ over $a = [a_{\nu_0,\cdots,\nu_n}]_{\nu_0+\cdots+\nu_n=\delta} \in \mathbb{P}_{\binom{n+\delta}{\delta}-1}$ is the hypersurface $X^{(a)}$.

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- *Slanted* means not tangential to a fiber.

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Clemens, H.: Curves on generic hypersurfaces, *Ann. Ec. Norm. Sup.* **19**(1986), 629–636.

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- In other words, certain dependence on a is transferred to dependence on z.

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- The space $J_{n-1}^{\mathrm{vert}}(\mathcal{X})$ of vertical $(n-1)$-jets of $\mathcal{X}$ is defined by $f = df = \cdots = d^{n-1}f = 0$ in $\left(J_{n-1}\left(\mathbb{P}_n\right)\right) \times \mathbb{P}_{\binom{n+\delta}{\delta}} - 1$
- with the coefficients $a_{\nu_0 \cdots \nu_n}$ of f regarded as constants when forming $d^j f$.
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- That is,

$$T_{J_{n-1}^{\mathrm{vert}}(\mathcal{X})} \otimes \mathcal{O}_{\mathbb{P}_n}(c_n) \otimes \mathcal{O}_{\mathbb{P}_{\binom{n+\delta}{\delta}-1}}(c_n')$$

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- with $0 < \theta_0 < 1$, $0 < \theta < 1$,
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- Every time Lie differentiation by slanted vector field is used, vanishing order of constructed jet differential at infinity is decreased.
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- In other words,

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- Here the notation for the Wronskian

$$\mathrm{Wron}(\eta_1, \cdots, \eta_\ell)$$

for jet differentials $\eta_1, \cdots, \eta_\ell$ on a complex manifold Y is used to mean

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$$\det\left(d^{\lambda-1}\eta_j\right)_{1\leq\lambda,j\leq\ell}=\sum_{\sigma\in S_\ell}(\operatorname{sgn}\sigma)\,\eta_{\sigma(1)}\left(d\eta_{\sigma(2)}\right)\cdots\left(d^{\ell-1}\eta_{\sigma(\ell)}\right)$$

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  in a neighborhood U of a point when F_j is nowhere zero on U for j not equal to any of the indices $\nu_1,\cdots,\nu_n$.

Second Main Theorem for Jet Differential with Slowly Moving Targets

- ASSUMPTION: Let $S \subset \mathbb{P}_N$ be a complex algebraic manifold and $\mathcal{X} \subset \mathbb{P}_{\hat{n}} \times S$ be a complex algebraic manifold.

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- Let $D = D_1 + \cdots + D_p$ and $E = E_1 + \cdots + E_q$,
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- CONCLUSION: Then

$$qT\left(r, \varphi, L + \pi^{-1}(L_S)\right) \leq N(r, \varphi, D) + o\left(T\left(r, \varphi, L + \pi^{-1}(L_S)\right)\right) \quad \|.$$

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